Hybrid discrete solitons

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The existence and stability of discrete solitons in waveguide arrays exhibiting a linear variation of the effective index and a Kerr nonlinearity is studied. We find that the resonant coupling of the conventional discrete soliton to a linear Wannier-Stark state does not entail soliton decay. We rather observe the formation of a bound state where the Wannier-Stark state gets nonlinearly modified. This results in an infinite number of isolated branches of hybrid discrete solitons.

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I. INTRODUCTION

Because of its relevance in many areas of science, nonlinear dynamics in discrete systems is well investigated. Prominent examples are molecular chains [1], low-dimensional crystals [2], chains of Josephson junctions [3], antiferromagnetic materials [4], and optical waveguide arrays. In particular, the latter system became very attractive as a convenient laboratory to visualize discrete dynamics [5].

The field dynamics in discrete systems differs from that in the respective continuous ones where even the linear features are considerably modified. For example, the character of diffraction and refraction of light beams in waveguide arrays depends strongly on the transverse wave vector component of the incident wave [6]. The overall behavior becomes even more complex if nonlinearity comes into play [7]. In waveguide arrays, already a simple Kerr nonlinearity produces a diversity of discrete soliton solutions [8,9]. The existence of dark soliton solutions for focusing nonlinearities [10] is just one example. Based on these exciting features that homogeneous arrays exhibit, it can be anticipated that linear and nonlinear dynamics in transversely inhomogeneous arrays may be even more diverse. Inhomogeneous chains of coupled oscillators (lattice) have attracted a great deal of interest since the beginning of the last century. In particular, a linear variation of the eigenfrequencies of local oscillators became a standard problem. Zener investigated electrons in lattices with an applied dc field and found that the inhomogeneity gives rise to the formation of localized eigenstates [Wannier-Stark states (WSSs)], the energy levels of which are equally spaced [11]. The appearance of this so-called Wannier-Stark ladder (WSL) in frequency space has its equivalent in a periodic motion in real space. But only recently these Bloch oscillations could be experimentally observed for electrons in semiconductor superlattices [12] for atoms in optical lattices [13] and photons in waveguide arrays [14,15]. However, the performance and observability of this phenomenon is often seriously deteriorated by nonlinear effects, i.e., nonlinearity can prevent the periodic field recovery and destroy the regular motion [15].

The aim of this paper is to disclose the interplay of linear and nonlinear localization in a waveguide array and to show that nonlinearly modified WSSs can form bound states with conventional discrete solitons. These novel localized states will be termed as hybrid discrete solitons and their stability will be probed.

Besides a linear variation of the properties of the constituents of the discrete system, other types of inhomogeneities have been extensively investigated in the literature. Similar to the formation of WSS, a stochastic variation of the properties of these constituents can result in the formation of localized linear eigenstates (Anderson localization) [16]. Likewise, if the finite size of the discrete system is taken into account, the spectral continuum modifies to a discrete set of localized modes. Previous research in nonlinear lattices [17,18] was aimed at understanding how these kinds of linear localization affect the existence and stability of nonlinear localized eigenstates, i.e., discrete breathers. Moreover, the linear properties of finite discrete systems with an additional linear variation of the lattice elements were studied [19].

II. THE MODEL

We will focus our investigation on waveguide arrays with a linear variation of the effective index of the individual waveguides, which are additionally endowed with a Kerr nonlinearity. The evolution of the mode amplitude $a_n(z)$ in the nth guide along the propagation direction z can be described by the normalized equations [15]

$$\left(i\frac{d}{dz}+\alpha n+|a_n|^2\right)a_n+a_{n+1}+a_{n-1}=0, \tag{1}$$

where the strength of the linear index variation is controlled by the parameter α. We note that similar dynamical equations describe the dynamics of Bose-Einstein condensates in a tilted periodic potential [20,21] and the motion of electrons in a crystal or a superlattice with an external dc electric field. Thus, provided that propagation distance z is replaced by time t, Eq. (1) corresponds to the nonlinear Wannier-Stark problem in one dimension.

Equation (1) is a modification of the nonintegrable discrete nonlinear Schrödinger equation and was used to model recent experiments in inhomogeneous nonlinear waveguide arrays [15]. It conserves power P and Hamiltonian H of the optical field as

$$P=\sum_n |a_n|^2, \tag{2a}$$

$$H=\sum_{n}\left(a_{n+1}^{*}a_{n}+a_{n+1}a_{n}^{*}+\alpha n|a_{n}|^{2}+\frac{1}{2}|a_{n}|^{4}\right). \tag{2b}$$

A similar model where the nonlinearity evokes a nonlinear coupling rather than a nonlinear index variation would be a generalized Ablowitz-Ladik equation [22–24]. Because of its integrability, the Ablowitz-Ladik equation is a well-investigated model, but unfortunately cannot be applied to waveguide arrays because it does not conserve the optical power.

As already mentioned, nonlinear dynamics is always governed by an interplay of linear and nonlinear properties of a specific system. Therefore we will start by summarizing the most important features of Eq. (1) in the low power regime, where it describes photonic Bloch oscillations. They manifest themselves by an oscillating propagation of discrete light beams in waveguide arrays with a linear modification of the effective index (Bloch arrays). Here we concisely summarize these linear features. An arbitrary linear eigenstate m of Eq. (1), i.e., the WSS, reads as

$$a_{n}^{m}(z)=u_{n}^{m}\exp(i\beta_{m}z). \tag{3}$$

Its stationary amplitude distribution

$$u_{n}^{m}=J_{m-n}(2/\alpha) \tag{4}$$

is localized but extends to a discrete plane wave of the homogeneous array for $\alpha\rightarrow 0$. Evidently, for increasing α, the localization increases too. All eigenstates with varying m have the same shape but are localized at different sites determined by the waveguide with number m.

The second important feature of solution (3) is the discrete spectrum of the longitudinal wave numbers $\beta_m=m\alpha$. This WSL of equidistant resonances sparsely fills the whole spectral range and gives rise to the recurrence phenomenon of Bloch oscillations.

Moreover, Eq. (1) implies a discrete translation symmetry,

$$\tilde{a}_{n}(z)=a_{n-\Delta}\exp(i\Delta\alpha z), \tag{5}$$

where every solution can be shifted by Δ guides by adding a wave number of $\Delta\alpha$. Symmetry relation (5) is conserved in the nonlinear case.

In addition, the dynamics is invariant with respect to a change of the sign of the Kerr nonlinearity. It can be shown that together with the transformation

$$a_{n}'(z)=(-1)^{n}a_{-n}^{*}(z),$$

this would just flip the solutions around and introduce a phase jump of π between adjacent guides.

III. DISCRETE SOLITON SOLUTIONS

In looking for soliton solutions of Eq. (1) in an inhomogeneous array as u_n^m in Eq. (3), i.e., discrete solitons or breathers, it is useful to identify some general properties of this basic equation.

Any motion of localized solutions across the waveguide array with a constant velocity would result in a monotonous increase or decrease of the Hamiltonian, thus violating conservation law (2). Hence, in the discrete system only resting solutions have to be considered. By contrast, for the equivalent continuous system, a transverse gradient of the refractive index would cause every solution to drift sidewards.

Because of phase symmetry one can get rid of a fast varying phase as in Eq. (3). Therefore the respective solutions become stationary. This has two main consequences. First, the whole spectrum is shifted; and second, no higher harmonics appear. This means that only the resonance of the fundamental wave number of the soliton with the linear waves is critical for the existence of the soliton solution.

The width of the linear WSSs depends on the detuning parameter α. For $\alpha=0$, the eigenfunctions are infinitely wide. Since for a finite power the nonlinearity only allows for a local modification of the linear system, the wave number of the soliton must not be situated within the linear spectrum. Therefore $\beta>2$ is required in Eq. (3) for $\alpha=0$ [25]. In contrast, for $\alpha\neq 0$, the unbound and discrete spectra of the Wannier-Stark problem allow for the existence of discrete solitons in the gaps of the WSL. Hence, unlike the $\alpha=0$ case, the spectrum of the nonlinear solutions will be somehow disrupted. But since for $\alpha\neq 0$ the linear eigenfunctions are localized, the linear spectrum can be locally modified by the nonlinearity with finite energies. Therefore the solitons can attain an arbitrary wave number for $\alpha\neq 0$. But one might observe some effects caused by the resonances with the linear spectrum.

The existence of localized WSSs has another consequence for the soliton solutions for vanishing power. Because the WSSs are localized for $\alpha\neq 0$ and low power nonlinear solutions must emanate from the linear eigenstates, soliton solutions emanate with a finite width from the WSSs. This is in contrast to the case $\alpha=0$, where the linear eigenfunctions are of infinite width, having the consequence that the soliton width, approaches infinity if $\beta\rightarrow 2$.

From above discussion it is obvious that the features of discrete solitons are strongly affected by the linear potential, i.e., linear index variation. Hence, we expect that new classes of localized solutions will emerge, displaying peculiar properties.

Looking for particular nonlinear eigensolutions (3) of Eq. (1), we used an implementation of Newton's method and found two distinct types of solutions. First, the fundamental single-lobed discrete soliton [8,9] of the homogeneous problem ($\alpha=0$) exists also for $\alpha\neq 0$, provided that its wave number β lies above the wave number of the WSS spatially overlapping with the discrete soliton. For example, in an array with $\alpha=0.5$, a localized solution centered around site m [which resembles a single lobed discrete soliton of the homogeneous problem ($\alpha=0$)], requires $\beta>3.5$. The second type represents the nonlinear continuation of WSSs, which can thus be termed nonlinear WSSs. In Fig. 1 both types are displayed. Besides these two types of very regular solutions, we found a variety of other solutions with different topology, which will be skipped in the present context.

For a fixed parameter α, the single-lobed solitons form a

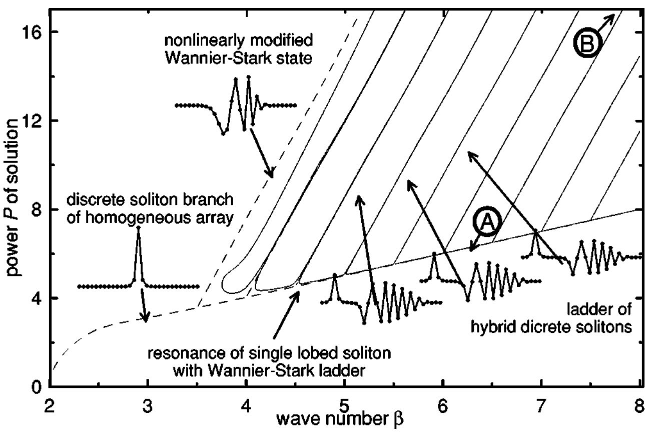

FIG. 1. Because of resonances with the linear spectrum, i.e., the WSL, the family of the single-lobed fundamental soliton, which is similar to the fundamental soliton of a homogeneous array, splits into a ladder of hybrid discrete solitons ($\alpha=0.5$).

family with the wave number β as the family parameter. However, unlike $\alpha=0$, this soliton branch breaks apart into several isolated areas. To understand this splitting, we take a closer look at the soliton shape displayed in Fig. 2. For some family parameters β, the soliton compares to a conventional discrete soliton of the homogenous array [$\alpha=0$, see Fig. 2(a)]. But one can imagine that the wave number β of the soliton centered around guide m can match the wave number β' of a nearby guide at site m' ($m'>m$) that is not detuned by nonlinearity but rather by the externally induced linear index variation ($\beta'-\beta=m'\alpha-\beta=0$). The resulting resonant coupling of the soliton to the guide m' excites the WSS $a^{m'}$ [see Eqs. (3) and (4)] that is centered around the guide m'. Usually, one would expect that the coupling of the solitary wave to the linear spectrum would lead to soliton decay.

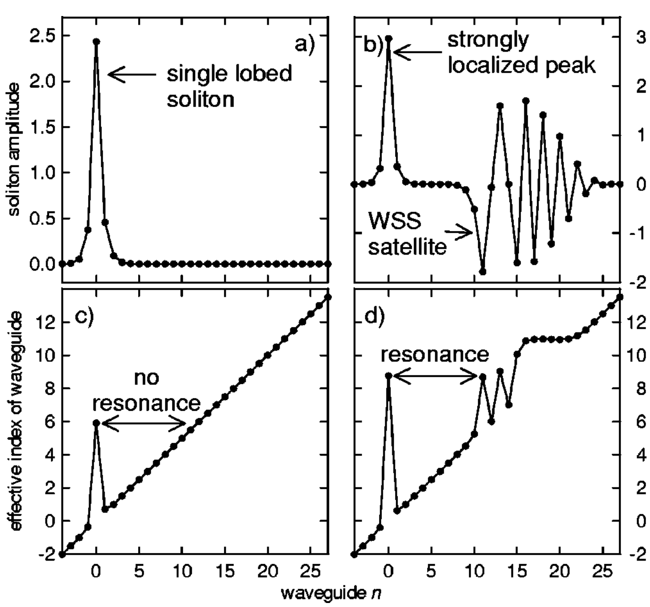

FIG. 2. Shape of soliton for (a) $\beta=6.25$, (b) $\beta=9$ (indicated in Fig. 1 as A and B); (c),(d) effective index of the nonlinearly detuned waveguides ($=\alpha n+|u_n^m|^2$) for solitons in (a),(b) ($\alpha=0.5$).

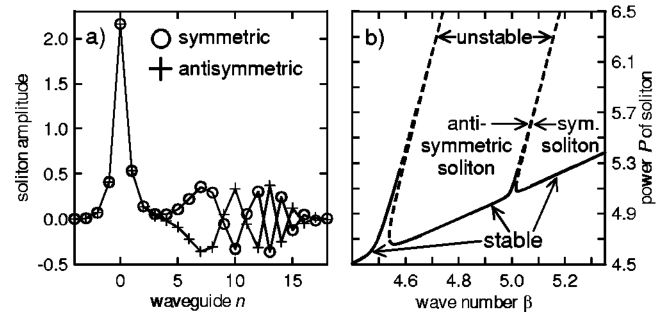

FIG. 3. Each resonance of the single-lobed soliton with the WSL gives rise to two hybrid soliton branches of opposite symmetry ($\beta=5.1$). The linear stability analysis in (b) reveals a destabilization of the single-lobed soliton by the resonances with the WSL ($\alpha=0.5$).

But instead, the single-lobed soliton and the emerging nonlinear WSS satellite form a different type of soliton, which we term hybrid soliton [see Fig. 2(b)]. Since for growing β these resonances appear in regular intervals, the conventional soliton branch is interrupted periodically. For each resonance with a WSS, two disconnected branches of hybrid solitons appear (see Fig. 3), which represent the two possible linear combinations between a single-lobed soliton and a WSS satellite. Tracking a hybrid soliton beyond the initial resonance, i.e., towards higher β, results in a growth of the satellite power. The nonlinearity modifies the guides in the area of the satellite such that they stay tuned to that of the original strongly localized part [see plateau in Fig. 2(d)]. It is interesting to note that even though the nonlinearity is focusing, the nonlinear WSSs delocalize for growing power.

If several soliton solutions coexist, one expects some of them to be unstable. Therefore we performed a linear stability analysis. After linearization around a soliton state, we determined the growth rate of respective perturbations. Like in the homogeneous array ($\alpha=0$), single-lobed solitons appear to be essentially stable. In contrast, nonlinear WSS are unstable over a wide range of the parameters α and β. Therefore, hybrid solitons mainly destabilize if the respective satellites become too large [see Fig. 3(b)]. Hence, each section of the branch of hybrid soliton solutions is sandwiched between two critical points, where instability sets in. We find the onset of a real-valued instability (simple exponential growth of an unstable eigenvector) at the point where the derivative of the soliton power with respect to the wave number changes sign. The stable area is limited on the other side by the onset of a complex-valued instability (oscillatory growth of an unstable eigenvector). Probing the dynamical stability by numerically integrating Eq. (1) supports the assumption that the origin of the instability lies in the WSS satellites. In Fig. 4, a hybrid soliton with strong satellite content was disturbed. While the nonlinear WSS satellite rapidly decays, the strongly localized part withstands the persistent distortions for a long time.

Another important fact can be observed in the propagation shown in Fig. 4. While the discrete soliton is destroyed by the dynamic instability, its power cannot spread substantially. Even after the decay of the soliton, most of the power stays

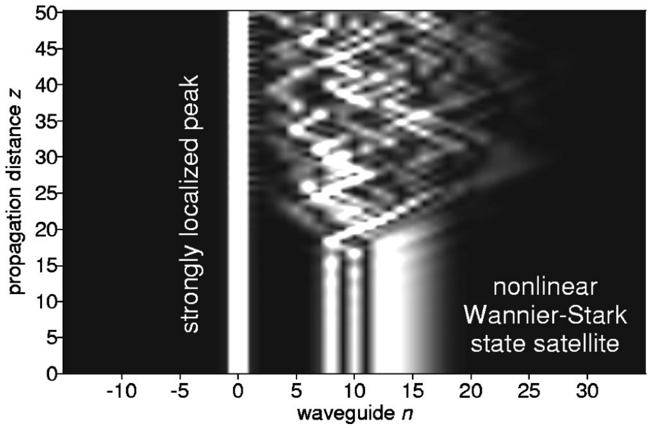

FIG. 4. Decay of an unstable hybrid soliton. The unstable WSS satellite rapidly decays, while the strongly localized soliton part is robust. The quasilinear radiation is still bounded by the linear index variation ($\alpha=0.5$, $\beta=7.0$).

confined to the area occupied by the soliton before the decay. This effect is one more consequence of the localization of the linear eigenstates, which also limits the linear spreading. Thus, only nonlinearity can transfer light to remote sites. But since nonlinear effects decrease for spreading light, this mechanism slows down rapidly. Furthermore, the conservation of the Hamiltonian (2b) must result in a rather symmetric spreading. For finite size arrays, the power dispersion would already stop when one end of the array is reached.

The localization of the linear waves also has consequences for the excitation dynamics of solitons. For systems with extended linear waves, all excess power of an imperfect excitation is radiated away by linear waves and only the soliton part remains localized. In contrast, the existence of localized WSSs leads to a different transient behavior. The situation is illustrated in Fig. 5(a) where a localized excitation gives birth to a soliton. But the excess light cannot escape from the soliton. Instead, it turns back and destroys it. It is interesting to note that the period after which the initial soliton is destroyed by the returning quasilinear radiation is a multiple of the Bloch oscillation period of the underlying

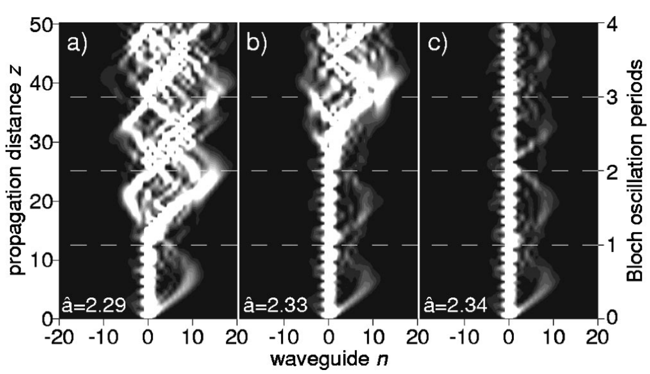

FIG. 5. Excitation dynamics of single-lobed discrete solitons [$a_0(0)=\hat{a}$, $a_{\neq 0}(0)=0$]. The excess light from the imperfect excitation returns to the discrete soliton after performing multiple Bloch oscillations. Below a certain threshold [(a) and (b)], the initially excited soliton is destroyed by this linear radiation ($\alpha=0.5$).

linear system. In Fig. 5(a), the linear radiation performs a single Bloch oscillation before it interacts with the soliton and destroys it. In Fig. 5(b), the excitation survives the first of these interactions. Thus the excitation of long living solitons requires a better matching of the input conditions to the stationary nonlinear solution than it is required for the corresponding homogeneous system. This is confirmed by the experimental results in Ref. [15] where the excitation of discrete optical solitons in linearly detuned waveguide arrays failed for conditions, which proved to be sufficient for the excitation of solitons in homogeneous arrays [5]. These findings are supported by numerical studies in Ref. [26], where it was found that self-trapping in a linearly detuned discrete system requires more power compared to the homogeneous one.

Putting our results in a broader context, we notice that the observed phenomena are much more general. Similar effects have been identified in various other systems with linear localization. A particular example are finite size lattices, where both the number and the extension of linear modes are limited [18]. In the free spectral range between different linear states, the so-called phantom breathers exist as nonlinear solutions with finite width. When a phantom breather solution is tracked towards another eigenfrequency and crosses the frequency of a linear state, the respective mode is excited. Consequently, a bifurcation diagram is observed, which is similar to that displayed in Fig. 3(b) [compare with Figs. 4 and 5(b) in Ref. [18]]. However, the resulting structures are much more complex than in the Bloch array. The reason is that the linear modes of a finite lattice have different shapes (not like the WSSs, which are basically identical) and that they usually overlap with respective localized nonlinear solutions in space. For the hybrid solitons, the localized part and the satellite are well separated. Furthermore, the solutions found in Ref. [18] strongly depended on the number of lattice sites. For hybrid solitons this is different, since already for a moderate number of waveguides ($N \gg 1/\alpha$) the WSL and the WSSs in the central part of the array correspond to those of an infinite system [19]. Therefore we did not observe any influence of the boundaries in our numerical simulations.

Disordered lattices, which seem to be rather distinct from the regular Bloch array, are another example, where linear states can be localized (Anderson localization) [16]. It was predicted that a fractal set of localized nonlinear solutions exists within the band of linear states. Every power increase causes these so-called intraband breathers to delocalize because they start to couple to various linear modes [27].

IV. CONCLUSIONS

In conclusion, we have shown that a family of single-lobed discrete solitons (which are similar to those known from conventional, homogeneous arrays) also exists in arrays with linear index variation. For distinct family parameters β, they become resonant with the linear spectrum of the Wannier-Stark states. While one would expect these resonances to annihilate the solitons, they rather lead to the formation of a new class of hybrid solitons. These hybrid solitons are formed as bound states of single-lobed discrete solitons and satellites, which correspond to the nonlinearly modified Wannier-Stark states of the system. The nonlinear Wannier-Stark states are essentially unstable and destabilize the originally stable single-lobed solitons near the resonance points. The dynamics of solitons is quite different from that in a homogenous array because of the localization of the linear waves. A clean excitation even of stable solitons is virtually impossible because excess energy cannot escape the soliton.

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